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COORDINATES IN DIFFERENTIAL GEODESY

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Preface

The following report is a written version of an abstract "Coordinates in Differential Geodesy" which was presented on August 23, 1991 at the XX General Assembly of the International Association of Geodesy in Vienna, Austria. The proceedings of the sessions on "General Theory and Methodology", Section IV were not published by the meeting and this report represents a slightly amplified version of my oral presentation.

The abstract addresses the various roles of coordinates in theoretical geodesy: determinative coordinates in classical geodesy, descriptive coordinates in the Marussi-Hotine formulation of differential geodesy, and finally our new notion of permissible coordinates in a reformulation of the Marussi-Hotine theory.

"COORDINATES IN DIFFERENTIAL GEODESY"

Joseph Zund

Summary:

This report is a written version of an abstract presented in August 1991, at the XX General Assembly of the International Association of Geodesy in Vienna. It concerns the various roles played by coordinates in differential geodesy, and proposes a new approach which offers the promise of significantly extending the ideas of Marussi and Hotine.

1. Introduction

Differential geodesy is concerned with the application of differential-geometric techniques to the curves and surfaces which occur in geodesy. It was originally devised by Antonio Marussi and Martin Hotine to give a three-dimensional formulation of geodesy, and to clarify its conceptual foundations, e.g. to *avoid* the *ab initio* dependence of its fundamental variables on a particular ellipsoidal reference system. In effect, Marussi and Hotine sought a reformulation of classical geodesy which would re-establish it as a branch of mathematical physics. Marussi laid down the basic mathematical and physical requirements of the theory (the Marussi conditions), he discovered the tensor of gravity gradients (the Marussi tensor), and introduced his local astronomical coordinate system; while Hotine gave a comprehensive presentation of the theory in his treatise (Hotine, 1969). The seminal source material which clearly indicates their ideas is contained in the monographs (Marussi, 1985) and (Hotine, 1991).

A basic problem in differential geodesy -- a problem which it inherits from differential geometry -- is the selection of appropriate local coordinate/reference systems. Such a selection is dependent on both mathematical *and* physical considerations, and since Hotine made an ambitious attempt to solve it in his treatise, it is called the *Hotine Problem*, (Zund, 1990). This problem is unsolved, and indeed it appears as difficult to properly pose as it is to solve it in practice. This abstract may be regarded as a progress report on our attempts to understand and properly formulate the Hotine Problem. This has required a re-thinking of the role of coordinates in differential geodesy and is based on the observation that logically such systems fall into two general classes which we call *determinative* and *descriptive coordinates*, respectively. Hence, coordinates belonging to different classes do not have the same properties and questions posed for one class need not be meaningful for the other class. In the following, for brevity we denote by x^r $(r = 1, 2, 3)$ an arbitrary coordinate system which may belong to either class.

2. Determinative Coordinates

Determinative coordinates are familiar from classical geodesy where they play an *active* role in the determination of the geopotential N as a solution of the Modified Laplace equation in a rotating system:

$$\Delta N = -2\tilde{\omega}^2 \tag{1}$$

where Δ is the 3-dimensional Laplace operator and $\tilde{\omega}$ is the (constant) angular velocity of the Earth. In (1) the unknown function N is a dependent variable, and the chosen x^r are independent variables. The choice of the x^r directly affects the prospects of solving (1) for N. In effect, N, is as general, or special, as the coordinates employed in determining it. An immediate concern in solving (1), or for that matter any of the partial differential equations occurring in classical geodesy, is whether the solutions are susceptible of a meaningful geometrical/physical interpretation. This is dramatically illustrated by the formal general solution of the Laplace equation

$$\Delta N = 0 \tag{2}$$

in a Cartesian coordinate system $y^r = (x, y, z)$:

$$N = f(ax + by + cz)\ ,$$

where f is an arbitrary smooth function and a, b, c are arbitrary complex constants subject to the condition

$$a^2 + b^2 + c^2 = 0\ .$$

This solution is global, since y^r is a global coordinate system on 3-dimensional Euclidean space E_3, however it has *no* immediate geometrical/physical significance.

3. Descriptive Coordinates

In differential geodesy as conceived by Marussi and Hotine, the function N is assumed to be *known*, and the analysis is then directed to describing the geometry and physics of the given geopotential field. In this case the x^r play a *passive* role whose *only* purpose is a descriptive one; hence such x^r may be called descriptive coordinates. Then (1) becomes an identity, or as one might more appropriately say when concerned with

describing the geometry and physics of the N-field, a *consistency equation* which couples N with the geometric/physical properties of the field. This geometry is formulated in terms of a 3-dimensional reference system, called a *vectorial* 3-*leg*, whose first and second vectors are associated with tangents to the equipotential surfaces, while the third vector is normal to these surfaces and consequently tangent to their plumblines. The dual reference system is a *Pfaffian* 3-*leg* and hence, the entire situation can be described in differential-geometric language without the *a priori* selection of a particular local coordinate system. Indeed, by use of the leg calculus, one resolves all quantities along the leg systems and so the principal variables/quantities appearing in the theory are *coordinate-free*. Since these quantities depend only on the choice of the 3-leg system they are called *leg coefficients*. The term *leg* is a literal translation of the German word *Bein* and was introduced by the Viennese geometers Duschek and Mayer (1930). It has been frequently employed in mathematical/theoretical physics, e.g. in general relativity, and more recently Grafarend (1986) has suggested its use in differential geodesy.

For example, it can be shown that (1) is replaced by the Bruns equation

$$N_{/3/3} - 2HN_{/3} = -2\tilde{\omega}^2 \tag{3}$$

where "/" denotes a leg (directional) derivative along the plumbline, and H is the mean curvature of the equipotential surfaces. Upon introducing the local gravity $n := N_{/3}$ (in Hotine notation), the above equation becomes

$$n_{/3} - 2Hn = -2\tilde{\omega}^2 \tag{4}$$

which specifies the change in gravity along the plumblines. Equation (4) is a consistency requirement which the components of the third leg vector ν^r, H, and $\tilde{\omega}$ must satisfy. If, as Hotine does throughout his treatise, one imposes the Marussi Ansatz $x^3 := N$ with (x^1, x^2) being local surface parameters on the equipotential surfaces, then (4) reduces to merely

$$\nu^3{}_{/3} - 2H\nu^3 = -2\tilde{\omega}^2 . \tag{5}$$

Thus, the distinction between dependent and independent coordinate variables is blurred, and the components of the leg variables λ, μ, ν become the basic variables. The local coordinates x^r are related to the contravariant components of the leg variables by the leg differential equations

$$x^r_{/1} := \lambda^r, \quad x^r_{/2} := \mu^r, \quad x^r_{/3} := \nu^r. \tag{6}$$

The full set of leg equations, which constitute the consistency requirements describing how the physical quantities N, $\varkappa$ and $\tilde{\omega}$ are coupled to the leg coefficients defining the geometry of the N-field, are called the *Hotine-Marussi equations* (although not all of these equations explicitly appear in their work, since neither of them employed more than a rudimentary version of the leg calculus). The unknown contravariant/covariant components of the leg vectors may be chosen as desired whenever they satisfy the Hotine-Marussi equations.

4. Hotine's Descriptive Coordinate Systems

In his treatise, Hotine endeavored to solve the Hotine Problem, i.e. construct local coordinate systems subject to the general requirements of the Marussi Conditions (which he did not formally state or endorse). Based on the Marussi Ansatz he proposed a hierarchy of five systems:

(ω, ϕ, N)-system (Chapters 12, 13, 14);

(x^1, x^2, N) normal-system (Chapter 15);

triply-orthogonal system (Chapter 16);

(ω, ϕ, λ)-system (Chapter 17);

symmetrical (ω, ϕ, λ)-systems (Chapter 18).

The triply-orthogonal system may be discarded by virtue of the falsity of the Hotine Conjecture (see Zund and Moore, 1987), and it turns out that the (ω, ϕ, λ)-systems are highly restrictive and can simply be regarded as specializations of the (ω, ϕ, N) and (x^1, x^2, N) normal-systems. The (ω, ϕ, N)-system is essentially the local astronomical system of Marussi, while the (x^1, x^2, N) normal-system is entirely due to Hotine. Both

of these systems can be characterized in terms of the contravariant components of the leg vectors as follows:

$$\lambda^r = (\lambda^1, \lambda^2, 0)\,,\ \ \mu^r = (\mu^1, \mu^2, 0)\,;$$

with

$$\nu^r = (\nu^1, \nu^2, n) \text{ in the } (\omega, \phi, N)\text{-system}\,,$$

$$\nu^r = (0, 0, n) \text{ in the } (x^1, x^2, N) \text{ normal-system.}$$

The (ω, ϕ, N) and (x^1, x^2, N) normal-systems are quite different and the former may be regarded as a specialization of the latter, but not conversely. It is possible to develop both from a more general notion which I call the *sistema finale*, or F-system. Using this system it can then be shown that the Marussi Ansatz involves no loss of generality since under a local coordinate transformation $x^r \mapsto \bar{x}^r$, one may choose

$$\bar{x}^1 = x^1\,,\ \ \bar{x}^2 = x^2\,,\ \ \bar{x}^3 = F$$

with the subsequent *ad hoc* identification of the arbitrary smooth function F with the geopotential N. Thus, the more general F-system provides a basic framework which includes both the (ω, ϕ, N) and (x^1, x^2, N) *normal-systems as limiting cases.* It might be noted that in a sense for descriptive systems, the question of the generality of the systems is a moot issue: a chosen x^r lacks generality only to the extent that they fail to provide a description, which is convenient, or otherwise, of a particular geometrical/physical field property.

Thus, the Marussi-Hotine approach to differential geodesy essentially contains two descriptive local coordinate systems, which as noted above are structurally different. For example, the (ω, ϕ, N)-system is applicable only to a single equipotential surface, viz it is 'frozen' on the surface. More precisely, if one considers a family Σ of such surfaces then one can choose the longitude and latitude, i.e. (ω, ϕ), as local surface parameters on *any* surface S of Σ, then one can take the same (ω, ϕ) as surface parameters on a neighboring equipotential surface only when

(i) S is a developable surface, or

(ii) the congruence Γ of plumblines of S are straight lines.

Both (i) and (ii) are severe restrictions: (i) requires that the Gaussian curvature K of S is zero, so typically S would be a cylinder, cone, etc. in contrast to our expectations of $K > 0$ for a sphere, spheroid or ellipsoid; (ii) requires that the curvature χ of the plumblines vanish so Γ is a linear congruence, hence S could be a sphere but *not* a spheroid or an ellipsoid! Hotine was aware of (ii), and devised his isozenithal differentiation (see Chapter 14, of (Hotine, 1969)) to work around it. However, in view of (i) such a manuever is of dubious value. The frozen character of the (ω, ϕ, N)-system is an especially heavy impediment in Hotine's work since many of his calculations in the treatise are given in this system. or one step away from it. The (x^1, x^2, N) normal-system was explicitly constructed to provide a local coordinate system for which (x^1, x^2) can be taken as local parameters on the surfaces of Σ in such a manner that (x^1, x^2) do not change along Γ. The notion of such a system is intuitively appealing and seems to have originated with Hotine, but is not a standard piece of Gaussian differential geometry. The centerpieces of his theory are his *variational equations* which prescribe how the intrinsic and extrinsic properties of the surfaces of Σ vary along Γ. These equations are undeniably elegant and a real tribute to Hotine's ingenuity. However, it is not clear whether they lead to useful results. For example, they give *no clue* as to what the (x^1, x^2) are, or how to choose them, or even whether a non-trivial pair of such parameters exist. Indeed, the variational equations are very complicated and appear to be integrable in closed form only when Σ reduces to a family of concentric spheres, i.e. Γ is a linear congruence. This need not be a serious defect, since the proper role of these equations is that of consistency, not determining, equations; however, by merely appending them to the Hotine-Marussi equations one greatly increases the complexity and the number of these equations. Thus, neither the (ω, ϕ, N) nor the (x^1, x^2, N) normal-system is truly satisfactory, or really workable. Both are heavily restricted, and it is possible that the more general (x^1, x^2, N)-system as derived from our F-system theory will prove more successful. What is clear is that -- at present -- we do not possess a really non-trivial solution of the Hotine Problem.

The purely determinative/descriptive role of the x^r is extreme: in classical geodesy the geometric content of the theory may be difficult to determine and often appears to be contrived, while in differential geodesy -- in the Marussi-Hotine formulation -- one is faced with devising a natural description of the properties of the geopotential field. Both approaches may be termed 'extractive' theories where one is, in effect, challenged to extract the geometry from a physical situation. These formulations are perfectly valid and workable, but neither is completely satisfying. Perhaps this is what Hotine sensed but did not put his finger on, when in (Hotine, 1965) he wrote:

> "For some time I have had an uncomfortable feeling that there is something missing, either in the classical theory of gravitation itself, or in the mathematical handling of it."

5. Permissible Coordinate Systems

Quite distinct from the Marussi-Hotine descriptive approach one may consider the situation in differential geodesy from an alternate more robust viewpoint. We call this a *permissible* viewpoint in which one specifies certain of the dynamical and geometric quantities, e.g. the leg coefficients: k_1, k_2, t_1, γ_1, γ_2, σ_1, σ_2, ε, of Hotine (1969) and Zund (1990), and then asks whether the Hotine-Marussi equations are compatible with this choice. In this case, one now seeks the geopotential function N subject to certain dynamical/geometric conditions and (3), or (4), is no longer an identity but a determinative equation and the solution of this equation must be consistent with the full set of Hotine-Marussi equations which include the Lamé equations for the flatness of E_3. Depending on the prescribed conditions some of these Lamé equations may be identities while others are non-trivial determining equations for the unspecified leg coefficients. For example, one might seek a solution for N in which $K := k_1 k_2 - t_1^2 > 0$ and $\chi^2 = \gamma_1^2 + \gamma_2^2 > 0$, i.e. one having equipotential surfaces of non-negative Gaussian curvature and curved plumblines, subject to certain non-zero components of the 3-leg vectors $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, and $\boldsymbol{\nu}$. Given such a specification either the Hotine Problem is solvable or

insolvable, and both situations are interesting. Relative to the former, one has constructed a geopotential field having certain prescribed dynamical/geometric properties; while in the latter case, assuming that the resulting Hotine-Marussi system of equations are tractable, one may show that the prescribed conditions are inconsistent or that the conditions are inadequate, i.e. not sufficiently specified, to permit a solution. Obviously the former case is the most interesting, but the latter is valuable in that it may indicate how the Hotine Problem should (or should not) be formulated. This approach is said to be *permissible*, since it involves exhibiting only those solutions of N and the unspecified leg coefficients which are permitted by the assumed dynamical/geometric conditions. The final step of such a procedure would be to solve (6) for the resulting *permissible coordinates* x^r. Such a methodology would be an *active* rather than a *passive*, or 'extractive', approach to differential geodesy, and could lead to a more robust and vigorous formulation of the geometric aspects of theoretical geodesy.

Thus, we believe that differential geodesy is a live and vibrant theory, a theory which may well step beyond the goals set for it by the visionary efforts of Marussi and Hotine. Relative to it and its prospects for the future I would concur with the famous comment of Sir Winston Churchill:

> "Now this is not the end. It is not even the beginning of the end. But it is, perhaps, the end of the beginning."

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